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Generalized Willmore functionals and related variational problems

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Abstract

The purpose of this paper is to study the conformally invariant functionals of hypersurfaces in a Riemannian manifold and variational problems related to these functionals. A class of conformal invariants is presented and the variational problem related to this class of conformally invariant functionals is studied.

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1. Introduction

Conformal differential geometry is an important branch of differential geometry, initiated by H. Weyl and E. Cartan. One of significant results in this theory is a result of Weyl (cf. [15], [1] and [2]) stating that the conformal Weyl curvature tensor is a conformal invariant, and a Riemannian manifold of dimension >3 is conformally flat if and only if the conformal Weyl curvature tensor vanishes. In this paper, we investigate the conformal differential geometry of hypersurface of a Riemannian manifold. In the first part, we introduce some conformal invariants of a hypersurface which is called *generalized Willmore functional*. In the second part which is the main content of this paper, we consider the variational problems related to these functionals.

Let $x: M^n \to N^{n+1}$ be an $n(\geqslant 2)$-dimensional hypersurface immersed in a Riemannian $(n+1)$-manifold N^{n+1}, and let g_{ij} be the Riemannian metric tensor on M^n induced by the immersion x and h_{ij} the second fundamental tensor of M^n at x. Then the eigenvalues $\lambda_1, \dots, \lambda_n$ of the matrix (h_{ij}) relative to the matrix (g_{ij}), i.e., the roots of the equation $\det(h_{ij} - \lambda g_{ij}) = 0$ in λ, are called the principal curvature of M^n at x, and the rth mean curvature σ_r of M^n at x is defined by

$$\binom{n}{k}\sigma_r = \sum_{i_1<\cdots<i_r} \lambda_{i_1}\cdots\lambda_{i_r}, \quad r = 1, \dots, n, \tag{1.1}$$

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where $\binom{n}{k}$ is a binomial coefficient. For convenience, we define $\sigma_0 = 1$.

All conformal mappings of N^{n+1} form a group, called conformal group of N^{n+1}. A quantity on M^n is called a conformal invariant if it is invariant under the conformal group of N^{n+1}. In the first part of this paper, we prove the following theorem.

Theorem 1.1. *Let $x : M^n \to N^{n+1}$ be a hypersurface immersed in a Riemannian $(n+1)$-manifold N^{n+1}, then for any integer r, $2 \leqslant r \leqslant n$, the functional*

$$W_r(M) = \begin{cases} \int_M Q_r^{n/r}\, dM, & r < n \text{ and is odd}, \\ \int_M |Q_r|^{n/r}\, dM, & r < n \text{ and is even}, \\ \int_M Q_r\, dM, & r = n, \end{cases} \tag{1.2}$$

where

$$Q_r = \sum_{k=0}^{r} (-1)^{k+1} \binom{r}{k} \sigma_1^{r-k} \sigma_k, \tag{1.3}$$

is a conformal invariant of M^n in N^{n+1}.

Remark 1.1. For $n = 2$ and N is an Euclidean space E^3, the result is due to W. Blaschke [3]. Moreover, for a compact oriented surface M^2 in E^3 from the Gauss–Bonnet formula it follows that $\int_M \sigma_1^2\, dM$ is a conformal invariant of M^2 in E^3 (cf. [16] and [6]) and the famous Willmore Conjecture says that if the genus of M^2 is 1, then $\int_M \sigma_1^2\, dM \geqslant 2\pi^2$ (cf. [17]), i.e. $W_2(M) \geqslant 2\pi^2$. The extremal surfaces of the functional W_2 are called Willmore surfaces. There has been important progress on Willmore surfaces in recent years. For instance, Bryant observed duality of Willmore surfaces [5]. Pinkall and his collaborators used quaternions and established a new function theory with application to Willmore surfaces [4]. For a general n and $r = 2$ in (1.2), the functional

$$W_2(M) = \int_M \left(\sigma_1^2 - \sigma_2\right)^{n/2} dM \tag{1.4}$$

is called *Willmore functional* (for a submanifold with higher codimensional $W_2(M)$ can be similarly defined) and B.Y. Chen proved that W_2 is a conformal invariant [6]. In 1998, Chang Ping Wang established the conformal differential geometry of submanifold [14]. In this significant theory, Wang present a system of complete conformal invariants in which $(\sigma_1^2 - \sigma_2)^{n/2}\, dM$ is taken as the volume element of the submanifold.

In case that N^{n+1} is Euclidean space E^{n+1}, Theorem 1 is due to Chuan-Chih Hsiung and John J. Levco [11]. It is well known [8] that every conformal mapping on a Euclidean space E can be decomposed into a product of similarity transformations and inversions. Hence, Hsiung and Levco used the properties of the inversion in Euclidean space E to prove their theorem (cf. [9], [10] and [11]). Our Theorem 1.1 above shows that, for a general Riemannian manifold N^{n+1}, W_r is also a conformal invariant.

In the second part of this paper we derive the Euler–Lagrange equation for critical values of W_r, for the hypersurface in space form (Theorem 3.1 in Section 3). A hypersurface in N^{n+1} is called W_r-conformal minimal if it is a solution of the equation. For $n = 3$, some interesting examples of W_3-conformal minimal hypersurfaces are given in the second part of this paper.

2. Some conformal invariants and the proof of Theorem 1.1

In this section we recall some elementary facts on the conformal differential geometry of a hypersurface. Let M^n be a hypersurface isometrically immersed in a Riemannian manifold N^{n+1} with metric g. Let $\{e_1, \ldots, e_{n+1}\}$ be a local orthonormal frame field on N^{n+1}, such that when it is restricted on M^n, $\{e_1, \ldots, e_n\}$ is tangent to M and e_{n+1} is a normal vector field of M. We denote the dual frame field of $\{e_1, \ldots, e_{n+1}\}$ by $\{\omega_1, \ldots, \omega_{n+1}\}$. we adopt the following convention on the ranges of indices

$$1 \leqslant i, j, k, \ldots \leqslant n; \qquad 1 \leqslant A, B, C, \ldots \leqslant n+1$$

and the Einstein convention on summation of repeated indices. Then the structure equations of N^{n+1} are

$$d\omega_A = \omega_B \wedge \omega_{BA}, \qquad \omega_{AB} + \omega_{BA} = 0 \tag{2.1}$$

$$d\omega_{AB} - \omega_{AC} \wedge \omega_{CB} = -\frac{1}{2}\bar{R}_{ABCD}\omega_C \wedge \omega_D, \tag{2.2}$$

where ω_{AB} and R_{ABCD} are the connection form and curvature induced by the metric g of N^{n+1}, respectively. The structure equations of the hypersurface M^{n+1} can be written as follows:

$$d\omega_i = \omega_j \wedge \omega_{ji}, \tag{2.3}$$

$$d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2}R_{ijkl}\omega_k \wedge \omega_l, \tag{2.4}$$

$$\omega_{i,n+1} = h_{ij}\omega_j, \qquad h_{ij} = h_{ji}, \tag{2.5}$$

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} + \bar{R}_{ijkl}, \tag{2.6}$$

$$h_{ij,k} - h_{ik,j} = -\bar{R}_{i(n+1)jk}, \tag{2.7}$$

where $h_{ij,k}$ is defined by

$$h_{ij,k}\omega_k = dh_{ij} + h_{ik}\omega_{kj} + h_{kj}\omega_{ki}. \tag{2.8}$$

Now let $\tilde{g}$ be a metric which is conformal with the metric g. Then there is a smooth function ϕ on N^{n+1} such that

$$\tilde{g} = e^{2\phi}g. \tag{2.9}$$

Set $\tilde{e}_A = e^{-\phi}e_A$ and let $\{\tilde{\omega}_A\}$ be the dual field of frame of $\{\tilde{e}_A\}$, then $\{\tilde{e}_A\}$ is a local orthonormal frame field with respect to the metric $\tilde{g}$ and

$$\tilde{\omega}_A = e^{\phi}\omega_A. \tag{2.10}$$

Let $\tilde{\omega}_{AB}$ be the connection form induced by the metric $\tilde{g}$. Then from (2.1) and (2.10), we have

$$\tilde{\omega}_{AB} = \omega_{AB} + \phi_B\omega_A - \phi_A\omega_B, \tag{2.11}$$

where $\phi_A := g(e_A, \nabla\phi)$ and ∇ is the gradient operator of the metric g. In particular, we have

$$\tilde{\omega}_{ij} = \omega_{ij} + \phi_j\omega_i - \phi_i\omega_j, \tag{2.12}$$

$$\tilde{\omega}_{i,n+1} = \omega_{i,n+1} + \phi_{n+1}\omega_i. \tag{2.13}$$

Let $\tilde{h}_{ij}$ be the second fundamental form of M^n as a hypersurface of a Riemannian manifold $(N^{n+1}, \tilde{g})$, which means that $\tilde{\omega}_{i,n+1} = \tilde{h}_{ij}\tilde{\omega}_j$. From (2.13) we have

$$\tilde{h}_{ij} = e^{-\phi}(h_{ij} + \phi_{n+1}\delta_{ij}). \tag{2.14}$$

Let $\lambda_1, \ldots, \lambda_n$ and $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n$ be the eigenvalues of the matrix (h_{ij}) and $(\tilde{h}_{ij})$ at a point x of M^n, respectively, σ_r and $\tilde{\sigma}_r$ their rth mean curvatures, respectively. From (2.14) we have

$$e^{\phi}(\tilde{h}_{ij} - \tilde{\sigma}_1\delta_{ij}) = h_{ij} - \sigma_1\delta_{ij}. \tag{2.15}$$

Let

$$B_{ij} = \sigma_1\delta_{ij} - h_{ij},$$

then (2.15) is equivalent to

$$e^{\phi}\tilde{B}_{ij} = B_{ij}. \tag{2.16}$$

Let $\mu_1, \ldots, \mu_n$ and $\tilde{\mu}_1, \ldots, \tilde{\mu}_n$ be the eigenvalues of the matrix (B_{ij}) and $(\tilde{B}_{ij})$ at the point x, respectively. Then $\mu_i = \sigma_1 - \lambda_i$, and (2.16) implies

$$e^{\phi}\tilde{\mu}_i = \mu_i. \tag{2.17}$$

If we define the quantity $-Q_r$ as the rth mean curvature with respect to (B_{ij}), i.e.:

$$-\binom{n}{r} Q_r = \sum_{1\leqslant i_1<\cdots<i_r\leqslant n} \mu_{i_1}\cdots\mu_{i_r}, \tag{2.18}$$

then from (2.17) we see that

$$e^{r\phi}\tilde{Q}_r = Q_r. \tag{2.19}$$

We call Q_r the rth *conformal mean curvature*. The relations between conformal mean curvatures and Euclidean mean curvatures are given by the following lemma.

Lemma 2.1. *Let Q_r be the rth conformal mean curvature defined by* (2.18). *Then we have*

$$Q_r = \sum_{k=0}^{r} (-1)^{k+1}\binom{n}{k}\sigma_1^{r-k}\sigma_k. \tag{2.20}$$

Proof. Since

$$\begin{aligned}
-\binom{n}{r} Q_r &= \sum_{1\leqslant i_1<\cdots<i_r\leqslant n} (\sigma_1-\lambda_{i_1})\cdots(\sigma_1-\lambda_{i_r}) \\
&= \sum_{1\leqslant i_1<\cdots<i_r\leqslant n} \left[\sigma_1^r - (\lambda_{i_1}+\cdots+\lambda_{i_r})\sigma_1^{r-1} + (\lambda_{i_1}\lambda_{i_2}+\cdots+\lambda_{i_{r-1}}\lambda_{i_r})\sigma_1^{r-2} + \cdots + (-1)^r\lambda_{i_1}\cdots\lambda_{i_r}\right] \\
&= \sum_{1\leqslant i_1<\cdots<i_r\leqslant n}\sum_{k=0}^{r}(-1)^k\sigma_1^{r-k}\sum_{1\leqslant j_1<\cdots<j_k\leqslant r}\lambda_{i_{j_1}}\cdots\lambda_{i_{j_k}} \\
&= \sum_{k=0}^{r}(-1)^k\sigma_1^{r-k}\sum_{1\leqslant i_1<\cdots<i_r\leqslant n}\sum_{1\leqslant j_1<\cdots<j_k\leqslant r}\lambda_{i_{j_1}}\cdots\lambda_{i_{j_k}},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{1\leqslant i_1<\cdots<i_r\leqslant n}\sum_{1\leqslant j_1<\cdots<j_k\leqslant r}\lambda_{i_{j_1}}\cdots\lambda_{i_{j_k}} &= \binom{n-k}{r-k}\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\lambda_{i_1}\cdots\lambda_{i_k} \\
&= \binom{n-k}{r-k}\binom{n}{k}\sigma_k = \binom{n}{r}\binom{r}{k}\sigma_k,
\end{aligned}$$

we have (2.20). This completes the proof of the lemma. □

Let dM and $\widetilde{dM}$ be the volume elements with respect to the metric g and $\tilde{g}$. Then we have

$$\widetilde{dM} = e^{n\phi}\,dM. \tag{2.21}$$

From (2.16), (2.20) and (2.21) we see that Theorem 1.1 holds.

Remark 2.1. Making use of (2.16) and using similar methods, one can construct other conformal invariants. But W_r is one of the most natural conformal invariants.

3. The variation for the functional W_r

Let $X : M \times R \to N^{n+1}$ be a smooth variation of x such that $X(\cdot, t) = x$ and $dx_t(T_pM) = dx(T_pM)$ on the boundary ∂M of M, for each (small) t. We call such variation an *admissible variation* of x. We note that the two boundary conditions for an admissible variation disappear if $\partial M = \emptyset$. For each t we denote by $\{e_i\}$ a local orthonormal basis for TM with respect to the metric induced by x_t, with dual basis $\{\omega_i\}$, and denote by e_{n+1} a unit normal vector field for x_t in N^{n+1}. Then $\{e_i, e_{n+1}\}$ is a moving frame of N^{n+1} along $M \times R$. Let $\bar{\omega}_A, \bar{\omega}_{AB}$ be the forms corresponding to this frame. If we write variational vector field of x in N^{n+1} by

$$\frac{\partial X}{\partial t} = \sum_i a_i e_i + f e_{n+1}, \tag{3.1}$$

then we have

$$\bar{\omega}_{n+1} = f\,dt, \quad \bar{\omega}_i = \omega_i + a_i\,dt. \tag{3.2}$$

Since $T^*(M \times R) = T^*M \oplus T^*R$, we have the decomposition

$$\bar{\omega}_{ij} = \omega_{ij} + b_{ij}\,dt, \tag{3.3}$$

$$\bar{\omega}_{i,n+1} = \omega_{i,n+1} + c_i\,dt, \tag{3.4}$$

where $\{f, a_i, b_{ij}, c_i\}$ are local functions on M with $b_{ij} = -b_{ji}$. We denote by d_M the differential operator on T^*M, then $d = d_M + dt \wedge \partial/\partial t$ on $T^*(M \times R)$. Using (2.1), (3.1) and (3.2) and comparing the terms in $T^*M \wedge dt$ we have

$$f_i = c_i - a_j h_{ji}, \tag{3.5}$$

$$\frac{\partial \omega_i}{\partial t} = \sum_j (a_{i,j} + b_{ij} - f h_{ij})\omega_j, \tag{3.6}$$

where f_i and $a_{i,j}$ are defined by $f_i\omega_i = d_M f$ and $a_{i,j}\omega_j = d_M a_i + a_j\omega_j$, respectively. From (3.6) we have

$$\frac{\partial}{\partial t}(\omega_1 \wedge \cdots \wedge \omega_n) = \left(\sum_i a_{i,i} - nf\sigma_1\right)\omega_1 \wedge \cdots \wedge \omega_n. \tag{3.7}$$

Using (3.4), (3.5) and (3.6), we have

$$\frac{\partial h_{ij}}{\partial t} = b_{ik}h_{kj} - h_{ik}b_{kj} + f_{i,j} + a_k h_{kij} + f\left(h_{ik}h_{kj} + \bar{R}_{i(n+1)j(n+1)}\right) + \bar{R}_{i(n+1)jk}a_k, \tag{3.8}$$

in particular:

$$n\frac{\partial \sigma_1}{\partial t} = \Delta f + (S + nc)f + \left(\sum_i h_{ki,i} + \bar{R}_{(n+1)k}\right)a_k. \tag{3.9}$$

Let $A = (h_{ij})$ and $T_{(k)}$ be the kth *Newton transformation* which is defined by

$$T_{(k)} = s_k I - s_{k-1}A + \cdots + (-1)^{k-1}s_1 A^{k-1} + (-1)^k A^k, \quad k = 0, 1, \ldots, n, \tag{3.10}$$

where $s_k = \binom{n}{k}\sigma_k$. The Newton transformations can be inductively defined by

$$T_0 = I, \qquad T_{(k+1)} = s_{k+1}I - T_{(k)}A. \tag{3.11}$$

The Hamilton–Cayley theorem implies that $T_{(n)} = 0$. The following equation is called Newton's formula

$$(k+1)S_{k+1} = \mathrm{Trace}(AT_{(k)}). \tag{3.12}$$

Using formula (3.12), Reilly proved the following lemma (cf. Lemma A in [18]).

Lemma 3.1.

$$\binom{n}{k}\frac{\partial \sigma_k}{\partial t} = T_{(k-1)ij}\frac{\partial h_{ij}}{\partial t}, \quad k = 1, \ldots, n. \tag{3.13}$$

If N^{n+1} is of constant curvature c, then we have $\bar{R}_{i(n+1)jk}=0$, $\bar{R}_{i(n+1)j(n+1)}=c\delta_{ij}$ and $h_{ij,k}-h_{ik,j}=0$. Hence, from (3.8), (3.9) and (3.13) we have

$$\begin{aligned}\frac{\partial Q_r}{\partial t}&=-r\sigma_1^{r-1}\frac{\partial\sigma_1}{\partial t}+(-1)^{r+1}\frac{\partial\sigma_r}{\partial t}+\sum_{k=1}^{r-1}(-1)^{k+1}\binom{r}{k}\left[(r-k)\sigma_1^{r-k-1}\frac{\partial\sigma_1}{\partial t}\sigma_k+\sigma_1^{r-k}\frac{\partial\sigma_k}{\partial t}\right]\\&=\sum_{k=1}^{r-1}(-1)^{k+1}\binom{r}{k}\frac{r-k}{n}\sigma_1^{r-k-1}\sigma_k\Delta f+\sum_{k=2}^{r}(-1)^{k+1}\binom{r}{k}\binom{n}{k}^{-1}\sigma_1^{r-k}T_{(k-1)ij}f_{,ij}\\&\quad+\left[\sum_{k=1}^{r-1}(-1)^{k+1}\binom{r}{k}\frac{r-k}{n}\sigma_1^{r-k-1}\sigma_k(S+nc)\right.\\&\quad\left.+\sum_{k=2}^{r}(-1)^{k+1}\binom{r}{k}\binom{n}{k}^{-1}\sigma_1^{r-k}T_{(k-1)ij}(h_{il}h_{lj}+\bar{R}_{i(n+1)j(n+1)})\right]f\\&\quad+\left[\sum_{k=0}^{r-1}(-1)^{k+1}\binom{r}{k}(r-k)\sigma_1^{r-k-1}\sigma_k(\sigma_1)_{,l}+\sum_{k=1}^{r}(-1)^{k+1}\binom{r}{k}\binom{n}{k}^{-1}\sigma_1^{r-k}T_{(k-1)ij}h_{ij,l}\right]a_l,\end{aligned}\tag{3.14}$$

where S denotes the square of the second fundamental form.

In the following we assume $r(<n)$ to be odd, or $r=n$, or $r(<n)$ to be even and Q_r semi-positive definite. Under the assumptions, W_r can be written as

$$W_r=\int_M Q_r^{\frac{n}{r}}\,dM.$$

From (3.7) and (3.14) we have

$$\begin{aligned}\frac{\partial W_r}{\partial t}&=\int_M\frac{n}{r}Q_r^{\frac{n-r}{r}}\frac{\partial Q_r}{\partial t}\omega_1\wedge\cdots\wedge\omega_n+Q_r^{\frac{n}{r}}\frac{\partial}{\partial t}(\omega_1\wedge\cdots\wedge\omega_n)\\&=\int_M\frac{n}{r}Q_r^{\frac{n-r}{r}}\left\{\sum_{k=1}^{r-1}(-1)^{k+1}\binom{r}{k}\frac{r-k}{n}\sigma_1^{r-k-1}\sigma_k\Delta f+\sum_{k=2}^{r}(-1)^{k+1}\binom{r}{k}\binom{n}{k}^{-1}\sigma_1^{r-k}T_{(k-1)ij}f_{,ij}\right.\\&\quad+\left[\sum_{k=1}^{r-1}(-1)^{k+1}\binom{r}{k}\frac{r-k}{n}\sigma_1^{r-k-1}\sigma_k(S+nc)\right.\\&\quad\left.+\sum_{k=2}^{r}(-1)^{k+1}\binom{r}{k}\binom{n}{k}^{-1}\sigma_1^{r-k}T_{(k-1)ij}(h_{il}h_{lj}+c\delta_{ij})\right]f\\&\quad\left.+\left[\sum_{k=0}^{r-1}(-1)^{k+1}\binom{r}{k}(r-k)\sigma_1^{r-k-1}\sigma_k(\sigma_1)_{,l}+\sum_{k=1}^{r}(-1)^{k+1}\binom{r}{k}\binom{n}{k}^{-1}\sigma_1^{r-k}T_{(k-1)ij}h_{ij,l}\right]a_l\right\}dM\\&\quad+\int_M Q_r^{\frac{n}{r}}\left(\sum_i a_{i,i}-n\sigma_1\right)f\,dM.\end{aligned}\tag{3.15}$$

Where we denote the volume element $\omega_1\wedge\cdots\wedge\omega_n$ by dM. Since Newton transformation T is of property

$$\sum_j T_{(k)ij,j}=0,$$

for an admissible variation, for any $\mu\in C^2(M)$, we have

$$\int_M\mu T_{(k)ij}f_{,ij}\,dM=\int_M fT_{(k)ij}\mu_{,ij}\,dM,\quad 0\leqslant k\leqslant n.\tag{3.16}$$

Hence

$$\begin{aligned}\frac{\partial W_r}{\partial t}\bigg|_{t=0} = \int_M \Bigg\{ &\Delta\left(Q_r^{\frac{n-r}{r}} \sum_{k=1}^{r-1}(-1)^{k+1}\binom{r}{k}\frac{r-k}{r}\sigma_1^{r-k-1}\sigma_k\right) \\ &+ \sum_{k=2}^{r} T_{(k-1)ij}\left(Q_r^{\frac{n-r}{r}}(-1)^{k+1}\frac{n}{r}\binom{r}{k}\binom{n}{k}^{-1}\sigma_1^{r-k}\right)_{,ij} \\ &+ Q_r^{\frac{n-r}{r}}(S+nc)\sum_{k=1}^{r-1}(-1)^{k+1}\binom{r}{k}\frac{r-k}{r}\sigma_1^{r-k-1}\sigma_k \\ &+ Q_r^{\frac{n-r}{r}}\sum_{k=2}^{r}(-1)^{k+1}\binom{r}{k}\binom{n}{k}^{-1}\frac{n}{r}\sigma_1^{r-k}T_{(k-1)ij}(h_{il}h_{lj}+c\delta_{ij}) - n\sigma_1 Q_r^{\frac{n}{r}}\Bigg\} f\,dM \\ &+ \int_M \Bigg\{ Q_r^{\frac{n-r}{r}}\frac{n}{r}\Bigg[\sum_{k=0}^{r-1}(-1)^{k+1}\binom{r}{k}(r-k)\sigma_1^{r-k-1}(\sigma_1)_{,l} \\ &+ \sum_{k=1}^{r}(-1)^{k+1}\binom{r}{k}\binom{n}{k}^{-1}T_{(k-1)ij}h_{ij,l}\Bigg]a_l + Q_r^{\frac{n}{r}}\sum_i a_{i,i}\Bigg\}\,dM. \end{aligned} \tag{3.17}$$

From Lemma 3.1 we see that $\sum_{ij} T_{(k-1)ij}h_{ij,l} = \binom{n}{k}\sigma_{k,l}$, one can easily check that

$$\begin{aligned}&\sum_{k=0}^{r-1}(-1)^{k+1}\binom{r}{k}(r-k)\sigma_1^{r-k-1}(\sigma_1)_{,l} + \sum_{k=1}^{r}(-1)^{k+1}\binom{r}{k}\binom{n}{k}^{-1}T_{(k-1)ij}h_{ij,l} \\ &\quad = \left[\sum_{k=0}^{r}(-1)^{k+1}\binom{r}{k}\sigma_1^{r-k}\sigma_k\right]_{,l} = (Q_r)_{,l}. \end{aligned} \tag{3.18}$$

Green's theorem and (3.18) show that the last integer of (3.17) vanishes. Thus, we see that $\frac{\partial W_r}{\partial t}|_{t=0} = 0$ if and only if

$$\begin{aligned}&\Delta\left(Q_r^{\frac{n-r}{r}}\sum_{k=1}^{r-1}(-1)^{k+1}\binom{r}{k}\frac{r-k}{r}\sigma_1^{r-k-1}\sigma_k\right) \\ &\quad + \sum_{k=2}^{r}T_{(k-1)ij}\left(Q_r^{\frac{n-r}{r}}(-1)^{k+1}\frac{n}{r}\binom{r}{k}\binom{n}{k}^{-1}\sigma_1^{r-k}\right)_{,ij} + Q_r^{\frac{n-r}{r}}(S+nc)\sum_{k=1}^{r-1}(-1)^{k+1}\binom{r}{k}\frac{r-k}{r}\sigma_1^{r-k-1}\sigma_k \\ &\quad + Q_r^{\frac{n-r}{r}}\sum_{k=2}^{r}(-1)^{k+1}\binom{r}{k}\binom{n}{k}^{-1}\frac{n}{r}\sigma_1^{r-k}T_{(k-1)ij}(h_{il}h_{lj}+c\delta_{ij}) - n\sigma_1 Q_r^{\frac{n}{r}} = 0. \end{aligned} \tag{3.19}$$

From (3.11) and (3.12) we see that

$$T_{(k-1)ij}(h_{il}h_{lj}+c\delta_{ij}) = \binom{n}{k}\big(n\sigma_1\sigma_k - (n-k)\sigma_{k+1} + ck\sigma_{k-1}\big), \tag{3.20}$$

and from (3.19) and (3.20), we get the following theorem.

Theorem 3.1. *Let $x : M^n \to R^{n+1}(c)$ be a hypersurface in a space form $R^{n+1}(c)$ and assume that $r = n$, or $r(<n)$ is odd, or $r(<n)$ is even and Q_r semi-positive definite. The first variational formula of the functional $W_r(M)$ is given by*

$$\begin{aligned}&\Delta\big(Q_r^{\frac{n-r}{r}}(Q_{r-1}+\sigma_1^{r-1})\big) \\ &\quad + \binom{n-1}{r-1}^{-1}\sum_{k=2}^{r}(-1)^{k+1}\binom{n-k}{r-k}T_{(k-1)ij}\big(Q_r^{\frac{n-r}{r}}\sigma_1^{r-k}\big)_{,ij}\end{aligned}$$

$$+ Q_r^{\frac{n-r}{r}}\left(n^2\sigma_1^2 - n(n-1)\sigma_2 + nc\right)\left(Q_{r-1} + \sigma_1^{r-1}\right) - n\sigma_1 Q_r^{\frac{n}{r}}$$
$$+ Q_r^{\frac{n-r}{r}}\binom{n-1}{r-1}^{-1}\sum_{k=2}^{r}(-1)^{k+1}\binom{n-k}{r-k}\binom{n}{k}\sigma_1^{r-k}\left(n\sigma_1\sigma_k - (n-k)\sigma_{k+1} + ck\sigma_{k-1}\right) = 0. \tag{3.21}$$

In particular, for $n = r$, we have

$$\Delta\left(Q_{n-1} + \sigma_1^{n-1}\right) + \sum_{k=2}^{n}(-1)^{k+1}T_{(k-1)ij}\left(\sigma_1^{n-k}\right)_{,ij}$$
$$+ \left(n^2\sigma_1^2 - n(n-1)\sigma_2 + nc\right)\left(Q_{n-1} + \sigma_1^{n-1}\right) - n\sigma_1 Q_n$$
$$+ \sum_{k=2}^{n}(-1)^{k+1}\binom{n}{k}\sigma_1^{n-k}\left(n\sigma_1\sigma_k - (n-k)\sigma_{k+1} + ck\sigma_{k-1}\right) = 0. \tag{3.22}$$

Remark 3.1. In particular, for $r = 2$ in (3.21), we recover the Willmore equation for hypersurfaces which was obtained by Pedit and Willmore [13](also see [7] and [12]):

$$2(n-1)\Delta\left(\rho^{n-2}\sigma_1\right) - 2T_{1ij}\left(\rho^{n-2}\right)_{,ij}$$
$$+ (n-1)\rho^{n-2}\left[2n(n-1)\sigma_1^3 - n(3n-4)\sigma_1\sigma_2 + n(n-2)\sigma_3\right] = 0. \tag{3.23}$$

Let $n = 2$ in (3.23) then the equation reduces to the original Willmore equation

$$\Delta(\sigma_1) + 2\sigma_1\left(\sigma_1^2 - \sigma_2\right) = 0.$$

Let $n = 3$ in (3.22), then

$$\frac{1}{3}\Delta\left(2\sigma_1^2 - \sigma_2\right) - \frac{1}{3}T_{1ij}(\sigma_1)_{,ij} + 4\sigma_1^4 - 7\sigma_1^2\sigma_2 + 2\sigma_2^2 + \sigma_1\sigma_3 = 0. \tag{3.24}$$

Definition. A hypersurface $x : M^n \to N^{n+1}$ is called W_r*-minimal* if it is a solution of Eq. (3.21).

We present some examples of W_3-minimal hypersurfaces in S^4 as follows. A trivial example is an umbilic sphere $S^3(r)$ in S^4(in this case $\sigma_1^2 = \sigma_2$ and $\sigma_2^2 = \sigma_1\sigma_3$). Proposition 3.1 below shows that there is no nontrivial W_3-minimal hypersurface in the class of compact Euclidean minimal hypersurfaces.

Proposition 3.1. *If a compact hypersurface M^3 in S^4 is Euclidean minimal and W_3-minimal, then it must be totally geodesic.*

Proof. From the assumption we have $\sigma_1 = 0$ and (3.24). Hence, from (3.24) we have $\Delta\sigma_2 = 6\sigma_2^2$, which implies $\sigma_2 = 0$ as M is compact. The equations $\sigma_1 = 0$ and $\sigma_2 = 0$ imply $\lambda_1 = \lambda_2 = \lambda_3 = 0$. □

Proposition 3.2. $x : M = S^1(a) \times S^2(b) \to S^4$ *is W_3-minimal if and only if*

$$M = S^1\left(\sqrt{2/3}\right) \times S^2\left(\sqrt{1/3}\right),$$

where $S^p(d)$ denotes a p-dimensional sphere with radius d.

Proof. We can choose a suitable local orthonormal frame field $\{e_i\}$ on M such that

$$h_{ij} = \begin{cases} b/a, & i = j = 1, \\ 0, & i \neq j, \\ -a/b, & i = j = 2, 3. \end{cases}$$

Let $x = a^2/b^2$, then we have $\sigma_1 = (1-2x)/3\sqrt{x}$, $\sigma_2 = (-2+x)/3$, $\sigma_3 = \sqrt{x}$. Putting these quantities into Eq. (3.24), we have

$$11x^4 - 34x^3 + 30x^2 - 10x - 4 = 0.$$

This equation has an unique solution $x = 2$ in the set of positive real numbers. As $a^2 + b^2 = 1$ we have $a = \sqrt{2/3}$, $b = \sqrt{1/3}$. $\square$

Corollary. *Those hypersurfaces in S^4 which are conformally equivalent to $S^1(\sqrt{2/3}) \times S^2(\sqrt{1/3})$ must be W_3-minimal.*

Remark. $S^1(\sqrt{2/3}) \times S^2(\sqrt{1/3})$ is also W_2-minimal in S^4.

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